E.g.,

$$\int_{-\infty}^{1} x^2 e^x\,dx = \lim_{a\to-\infty}\int_a^1 x^2 e^x\,dx$$

$$\int_a^1 \underset{\substack{\uparrow\\ u}}{x^2}\ \underset{\substack{\uparrow\\ \frac{dv}{dx}}}{e^x}\,dx$$

$u = x^2.\quad \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^x \quad v = e^x$

$$= x^2e^x\Big|_a^1 - 2\int_a^1 \underset{\substack{\uparrow\\ \frac{dv}{dx}}}{e^x}\ \underset{\substack{\uparrow\\ u}}{x}\,dx$$

rename $u = x \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^x \quad v = e^x$

$$= x^2e^x\Big|_a^1 - 2\left(xe^x\Big|_a^1 - \int_a^1 e^x\,dx\right)$$

$$= x^2e^x\Big|_a^1 - 2xe^x\Big|_a^1 + 2e^x\Big|_a^1$$

$$= e - a^2e^a - \cancel{2e} + 2ae^a + \cancel{2e} - 2e^a$$

$$= \left(-a^2+2a-2\right)e^a + e$$

Type $0\cdot\infty$

$$\int_{-\infty}^{1} x^2e^x\,dx = \lim_{a\to-\infty}\left(-a^2+2a-2\right)e^a + e$$

$$= \lim_{a\to-\infty}\frac{-a^2+2a-2}{e^{-a}} + e$$

L'Hôpital

$$= \lim_{a\to-\infty}\frac{-2a+2}{-ae^{-a}} + e$$

L'Hôpital

$$= \lim_{a\to-\infty}\frac{-2}{-e^{-a}+a^2e^{-a}} + e \qquad \underbrace{(a^2-1)}_{\infty}\underbrace{e^{-a}}_{\infty}$$

$$= 0 + e$$

$$= e$$

□.

Example 5.2. Compute $\displaystyle\int_0^{+\infty} \frac{dx}{(x+1)(3x+2)}$. $\displaystyle = \lim_{b\to\infty} \int_0^b \frac{1}{(x+1)(3x+2)}\,dx$

Solution.

$$\int_0^b \frac{1}{(x+1)(3x+2)} = \int_0^b \frac{3}{3x+2} - \int_0^b \frac{1}{x+1}.$$

$$\frac{a}{x+1} + \frac{b}{3x+2} = \frac{a(3x+2)}{(x+1)(3x+2)} + \frac{b(x+1)}{(3x+2)(x+1)} = \frac{a(3x+2)+b(x+1)}{(x+1)(3x+2)}$$

Hence

$$\int_0^b \frac{dx}{(x+1)(3x+2)} = \left(\ln|3x+2| - \ln|x+1|\right)\Big|_0^b$$

$$= \ln|3b+2| - \ln|b+1| - \ln|2| = \ln\frac{|3b+2|}{|b+1|} - \ln 2.$$

$1 = a(3x+2) + b(x+1)$

plug in $x=-1$

$1 = a\cdot(-1) \leadsto a = -1$

plug in $x = -\frac{2}{3}$

$1 = b\cdot\frac{1}{3} \to b = 3.$

Because

$$\lim_{b\to+\infty} \frac{|3b+2|}{|b+1|} = \lim_{b\to+\infty} \frac{|3b+2| \times \frac{1}{|b|}}{|b+1| \times \frac{1}{|b|}}.$$

$$\lim_{b\to+\infty} \frac{|3+\frac{2}{b}|}{|1+\frac{1}{b}|} = \frac{3}{1} = 3.$$

Therefore

$$\lim_{b\to+\infty} \int_0^b \frac{dx}{(x+1)(3x+2)} = \ln 3 - \ln 2.$$

■

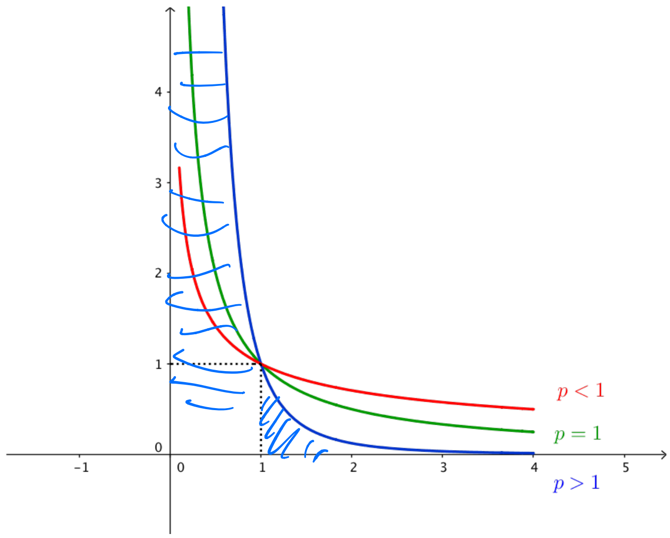
Exercise 5.1. Let $p > 1$. Prove that

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1}, & \text{if} \quad p > 1, \qquad \text{convergent} \\ +\infty, & \text{if} \quad 0 < p \le 1, \qquad \text{divergent}. \end{cases}$$

Remark. From the above exercise,

1. $\displaystyle\lim_{x\to+\infty} f(x) = 0 \quad \not\Rightarrow \quad \int_1^{+\infty} f(x)\,dx$ is convergent.
2. For all $p > 0$, $\dfrac{1}{x^p} \to 0$ as $x \to +\infty$. However, only for $p > 1$, $\dfrac{1}{x^p}$ decays fast enough to guarantee the total area $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\,dx$ is finite.

Remark. All the integration techniques can be applied, e.g. integration by substitution,...



Example 5.3. Compute $\int_{-\infty}^{1} xe^x\,dx$. (integration by parts)

Solution.

$$\int_{-\infty}^{1} xe^x\,dx = \lim_{a\to-\infty}\int_a^1 xe^x\,dx.$$

$$\int xe^x\,dx = \int x d(e^x) = xe^x - \int e^x\,dx = (x-1)e^x + C.$$

$$\begin{aligned}
\int_{-\infty}^{1} xe^x\,dx &= \lim_{a\to-\infty}(x-1)e^x|_a^1 \\
&= \lim_{a\to-\infty}(1-a)e^a \quad \infty\cdot 0 \quad \text{indeterminate form} \\
&= \lim_{a\to-\infty}\frac{1-a}{e^{-a}} \quad \frac{\infty}{\infty} \\
&= \lim_{a\to-\infty}\frac{-1}{-e^{-a}} \quad \text{L'Hôpital's rule} \\
&= 0.
\end{aligned}$$

■

Exercise 5.2. $\int_{-\infty}^{1} x^2e^x dx = e$

Example 5.4. Compute $\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx$. (integration by substitution)

$u = 1 + x^2$.

$\frac{du}{dx} = 2x$

Solution. Using the substitution $u = 1 + x^2$, we have

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{-1}{2(1+x^2)} + C.$$

u^2

$\int \frac{x\,dx}{(1+x^2)^2} = \int \frac{\frac{1}{2}du}{u^2}$

$= \frac{1}{2} \cdot \left(-\frac{1}{u}\right)$

Thus

$$\int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}$$

and

$$\int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}.$$

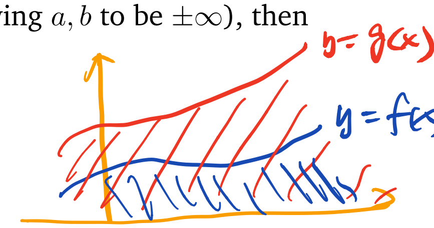
Hence

$$\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx + \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

$= \frac{1}{2}\left(-\frac{1}{u}\right)\Big|_{1+0^2}^{\infty} + \frac{1}{2}\left(-\frac{1}{u}\right)\Big|_{\infty}^{1}$

■

Fact: If $0 \le f(x) \le g(x)$ on the interval of integration (a, b) (allowing a, b to be $\pm\infty$), then



- If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.
- If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ diverges.

Example 5.5. Determine whether $\int_0^{\infty} x^n e^{-x} dx$ is convergent. $n > 0$

$x^n \ll e^{\frac{x}{2}}$ when x is very large.

$\int_0^b x^n e^{-x} dx + \int_b^{\infty} x^n e^{-x} dx$ b very large

finite

$\le \int_b^{\infty} e^{\frac{x}{2}} \cdot e^{-x} dx = \int_b^{\infty} e^{-\frac{x}{2}} dx$

convergent

E.g, $\int_1^{\infty} \frac{e^{-x}}{x} dx$. is convergent

because $\frac{e^{-x}}{x} \le e^{-x}$ for $x \ge a$

$\int_1^{\infty} \frac{e^{-x}}{x} dx \le \int_1^{\infty} e^{-x} dx$

convergent

so is also convergent

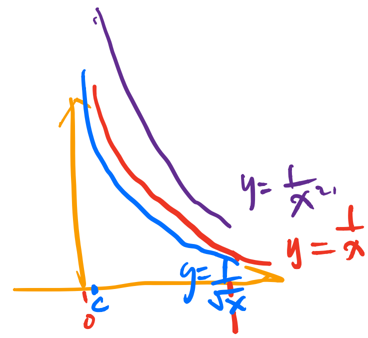
Definition 5.2 (Improper integrals of Type 2)**.** The improper integrals defined in Definition 5.1 has infinite intervals of integration, but the values of the integrand are finite on the intervals of the integration. We also generalize definite integrals where the integrand may go to $\pm\infty$ over the interval of integration.

Suppose that $f(x)$ is continuous on (a, b), but $\lim_{x\to b^-} f(x) = \pm\infty$. Then we define:

$$\int_a^b f(x)dx := \lim_{c\to b^-} \int_a^c f(x)dx.$$

Similarly, if $\lim_{x\to a^+} f(x) = \pm\infty$,

$$\int_a^b f(x)dx := \lim_{c\to a^+} \int_c^b f(x)dx.$$



Example 5.6. 1. $\displaystyle\int_0^1 \frac{1}{x^p}dx$:

$p > 0$

divergent when $p \geq 1$

convergent when $p < 1$

2. $\displaystyle\int_0^1 \frac{1}{\ln x}dx$

3. (mixed type) $\displaystyle\int_{-\infty}^1 \frac{1}{x^3}dx$

$$= \lim_{a\to-\infty} \int_a^{-1} \frac{1}{x^3}dx + \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^3}dx + \lim_{c\to 0^+} \int_c^1 \frac{1}{x^3}dx$$